REDUCIBLE VON NEUMANN GEOMETRIES

IN MEMORY OF MAURICE AUDIN

BY

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1. **Content of this paper.** Given any irreducible von Neumann geometry L and any complete Boolean algebra B we shall construct a particular reducible geometry $B(L)$ which has its centre isomorphic to B and is such that each Iwamura local component of $B(L)$ contains a sublattice isomorphic to L. If B is finite or if L is compact (in the dimension topology) each Iwamura local component is actually isomorphic to L.

We shall use a point-free generalization of the constructon given in [2, §6].

A von Neumann geometry (briefly, a geometry) L is a complemented modular lattice containing at least two elements which is complete and satisfies von Neumann's lattice continuity conditions [6, pp. 1, 2, Axioms I-IV].

Our construction of $B(L)$ is valid and yields a von Neumann geometry whenever (i) B is a Boolean algebra, (ii) L is a complemented modular lattice, and (iii) on L is given a real valued function $D(a)$ such that $0 \leqq D(a) \leqq 1$ for each $a \in L$, $D(0) = 0$, $D(1) = 1$, and $D(a \cup b) + D(a \cap b) = D(a) + D(b)$ for all $a, b \in L$. Thus $B(L)$ should be denoted more precisely as $B(L,D)$. However, as von Neumann showed [6, Part I, Chapter VII], if L is a von Neumann geometry and irreducible (as defined in [6, p. 3, Axiom VI]) such a dimension function exists and is unique.

2. **Preliminaries.** As in [2] we use the function $d(a,b) = D(a \cup b) - D(a \cap b)$ defined for all $a, b \in L$. We have: $d(a,a) = 0$, $d(a,b) = d(b,a) \geqq 0$ for all a,b in L (we do not exclude $d(a,b) = 0$ for $a \neq b$). As in [5, p. 102; 1, pp. 76–77], we have, successively,

$$\begin{aligned} d(a \cup c,\ b \cup c) + d(a \cap c, b \cap c) &= D(a \cup b \cup c) - D((a \cup c) \cap (b \cup c)) \\ &\quad + D((a \cap c) \cup (b \cup c)) - D(a \cap b \cap c) \\ &\leqq D(a \cup b \cup c) - D((a \cap b) \cup c) \\ &\quad + D((a \cup b) \cap c) - D(a \cap b \cap c) \end{aligned}$$

(since in every lattice, $(a \cup c) \cap (b \cup c) \geqq (a \cap b) \cup c$ and $(a \cap c) \cup (b \cap c) \leqq (a \cup b) \cap c$)

$$= D(a \cup b) + D(c) - (D(a \cap b) + D(c)) = d(a,b);$$

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$$
\begin{aligned}
d(a,b) &+ d(b,c)\\
&= d(a \cup b,b) + d(b,a \cap b) + d(b \cup c,b) + d(b, b \cap c)\\
&\geqq d(a \cup b \cup c, b \cup c) + d(b \cup c, b) + d(b, b \cap c) + d(b \cap c, a \cap b \cap c)\\
&= d(a \cup b \cup c, a \cap b \cap c)\\
&\geqq d(a \cup c, a \cap c) = d(a, c);
\end{aligned}
$$

$$
\begin{aligned}
d(a \cap c, b \cap d) &+ d(a \cup c, b \cup d)\\
&\leqq d(a \cap c, b \cap c) + d(b \cap c, b \cap d) + d(a \cup c, b \cup c) + d(b \cup c, b \cup d)\\
&\leqq d(a,b) + d(c,d).
\end{aligned}
$$

3. **Construction of** $B(L)$. A set of nonzero disjoint elements in B, $u = \{b\}$ with supremum $= 1$ will be called a *partition*. If u,v are partitions then uv denotes the partition consisting of all nonzero $b \cap c$ with $b \in u$, $c \in v$.

A function f defined for all $b \in u$ with values in L will be called a *partition-function*, more precisely, a *u-function*. If f is a u-function and g is a v-function then $f \cup g, f \cap g$ will denote the uv-functions with values $f(b) \cup g(c)$, respectively, $f(b) \cap g(c)$ for nonzero $b \cap c (b \in u, c \in v)$. We define $d(f,g)$ to be $\sup(d(f(b), g(c)) \mid b \in u, c \in v, b \cap c \neq 0)$. Clearly, if h is a w-function then $(f \cup g) \cup h = (f \cup h) \cup (g \cup h)$.

We shall call $f = \{f^n\}$ *fundamental* (or a fundamental sequence) if each f^n is a partition-function and $d(f^n,f^m) \to 0$ as $n, m \to \infty$. Because of the results of §2 it follows that if $f = \{f^n\}$ and $g = \{g^n\}$ are fundamental, then as $n,m \to \infty$, $d(f^n,g^m)$ converges to a limit, which we denote $d(f,g)$; $d(f,f) = 0$, $d(f,g) = d(g,f) \geqq 0$, $f \cup g \equiv \{f^n \cup g^n\}$ and $f \cap g = \{f^n \cap g^n\}$ are both fundamental, $d(f,g) = d(f \cup g, f \cap g)$, and if f, g, h, k are fundamental, then

$$
\begin{aligned}
d(f,g) &\leqq d(f,h) + d(h,g),\\
d(f \cap g, h \cap k) &\leqq d(f,h) + d(g,k),\\
d(f \cup g, h \cup k) &\leqq d(f,h) + d(g,k).
\end{aligned}
$$

If f,g are fundamental we write $f \equiv g$ if $d(f,g) = 0$ and $f \leqq g$ if $d(f \cup g,g) = 0$ (equivalently, $f \cup g \equiv g$). Clearly, $d(f,g) = d(f_1, g_1)$ if $f \equiv f_1$ and $g \equiv g_1$.

The relation $\equiv$ is easily seen to be an equivalence relation on the set of fundamental sequences; from now on we identify each fundamental sequence with its equivalence class. With this identification, we denote the set of fundamental sequences (or their equivalence classes) by $B(L)$.

Clearly $B(L)$ is ordered by the relation $f \leqq g$ defined above and $B(L)$ has a zero (namely, $f = \{f^n\}$ where for each n, $f^n(b)$ is defined for $b = 1 (\in B)$ with value $0 (\in L)$), and a unit (namely $f = \{f^n\}$ where for each n, $f^n(b)$ is defined for $b = 1 (\in B)$ with value $1 (\in L)$).

LEMMA 1. *If f,g are in $L(B)$ then $f \cup g$, $f \cap g$ are effective as* $\sup(f,g)$, $\inf(f,g)$, *respectively, and $L(B)$ is a modular lattice.*

Proof. $(f \cup g) \cup f = f \cup g$ so $f \cup g \geqq f$. Similarly $f \cup g \geqq g$. On the other hand, if h is fundamental and $h \geqq f$, $h \geqq g$ hold, then because of §2, $d(h \cup (f \cup g), h) \leqq d(h \cup f, h) + d(h \cup g, h) = 0 + 0 = 0$ so $h \geqq f \cup g$. Thus $f \cup g$ is effective as $\sup(f,g)$. Similarly $f \cap g$ is effective as $\inf(f,g)$.

LEMMA 2. *$L(B)$ is complemented.*

Proof. Let f be a given fundamental sequence. By using an equivalent fundamental sequence $\{f^n\}$ we may suppose (i): each f^n is a u_n-function for partitions u_n such that $b \in u_n$, $c \in u_{n+1}$, $b \cap c \neq 0$ implies $c \leqq b$, and (ii): $d(f^n, f^{n+1}) < 1/n^2$ for all n. Then we need only find u_n-functions g for $n = 1,2,\cdots$ with the properties $g^n(b) \cup f^n(b) = 1$, $g^n(b) \cap f^n(b) = 0$ for each $b \in u_n$ and $d(g^n, g^{n+1}) \leqq d(f^n, f^{n+1})$. This $g = \{g^n\}$ is necessarily fundamental and by Lemma 1, $\sup(f,g) = f \cup g = 1$, $\inf(f,g) = f \cap g = 0$, so g is a complement of f in $B(L)$.

Thus we need only show: for any $n \geqq 1$ and $b \in u_n$, given $a_1 = f^n(b)$, $a_2 = g^n(b)$ with $a_1 \cup a_2 = 1(\in L)$, $a_1 \cap a_2 = 0(\in L)$ and given $a_3 = f^{n+1}(c)$ for some $c \in u_{n+1}$ with $c \leqq b$, there exists $a_4 \in L$ such that $a_3 \cup a_4 = 1$, $a_3 \cap a_4 = 0$ and $d(a_2, a_4) = d(a_1, a_3)$. For then we can define $g^{n+1}(c) = a_4$.

As shown in [2, Lemma 6.9], a_4 can be chosen as

$$[a_2 - ((a_1 \cup a_3) \cap a_2)] \cup [a_1 - (a_1 \cap a_3)]$$

($[c - d]$, defined for $d \leqq c$, denotes any (fixed) relative complement of d with respect to c). With this a_4,

$$\begin{aligned} a_3 \cup a_4 &= [a_2 - ((a_1 \cup a_3) \cap a_2)] \cup a_1 \cup a_3 = a_2 \cup a_1 \cup a_3 = 1; \\ a_3 \cap a_4 &= a_3 \cap (a_1 \cup a_3) \cap a_4 = a_3 \cap [a_1 - (a_1 \cap a_3)] = 0; \\ d(a_2, a_4) &= D(a_2 \cup [a_1 - (a_1 \cap a_3)]) - D([a_2 - ((a_1 \cup a_3) \cap a_2)]) \\ &= D(a_2) + D(a_1) - D(a_1 \cap a_3) - D(a_2) + D((a_1 \cup a_3) \cap a_2) \\ &= D(a_1) - D(a_1 \cap a_3) + D(a_1 \cup a_3) + D(a_2) - D(a_1 \cup a_3 \cup a_2) \\ &= D(a_1 \cup a_3) - D(a_1 \cap a_3) + 1 - 1 = d(a_1, a_3). \end{aligned}$$

LEMMA 3. *$L(B)$ is complete.*

Proof. We need only show that for any given set of fundamental sequences $\{f_\alpha \mid \alpha \in I\}$, the supremum exists in $L(B)$. Since $L(B)$ is known to be a lattice we may replace $\{f_\alpha \mid a \in I\}$ by the set of all $\{f_s \mid s \subset I, s \text{ finite}\}$ where $f_s = \bigcup (f_\alpha \mid \alpha \in s)$. For each m we choose $m(s)$ by induction on m so that $m(s) > (m-1)(s)$ and $d(f_s^m, f_s^p) < 1/m$ for all $m, p \geqq m(s)$.

We shall now define a fundamental sequence $g = \{g^n\}$ and then verify that this g is effective as $\sup\{f_s\}$. We shall define g^n to be a u_n-function with partitions u_n such that $b \in u_n$, $c \in u_{n+1}$, $b \cap c \neq 0$ imply $c \leqq b$.

Let u_1 consist of the single element 1 and set $g^1(u_1) = 0$ $(\in L)$. This defines u_1 and g^1.

Suppose for some $n > 1$ that $u_1, \cdots, u_{n-1}, g^1, \cdots, g^{n-1}$ have been defined. Then for each $a \in u_{n-1}$ we shall choose certain nonzero disjoint $b \leqq a$ and define $g^n(b)$ for these b in such a way that the set of all chosen b (for all $a \in u_{n-1}$) will form a partition u_n and g^n will be a u_n-function.

We shall actually define n subsets of $\{b\}$, namely $a_n^1, \cdots, a_n^r, \cdots, a_n^n$ and define $g^n(b)$ for $b \in a_n^r$, as follows.

Choose a_n^1 to be a maximal set of nonzero disjoint $b \leqq a$ such that for each $b \in a_n^1$ there exist s, m, c with $m \geqq n(s)$, $f_s^m(c)$ defined and $b \leqq c$, and $D(f_s^m(c)) \geqq 1 - 1/n$, and set $g^n(b) = f_s^m(c)$.

If $a_n^1, \cdots, a_n^r$ have been defined for some $1 \leqq r < n$ then let

$$\bar{a} = a - \bigcup (b \mid b \in a_n^1 \cup \cdots \cup a_n^r)$$

and choose a_n^{r+1} to be a maximal set of nonzero disjoint $b \in B$ with $b \leqq \bar{a}$ such that for each $b \in a_n^{r+1}$ there exist s, m, c with $m \geqq n(s)$, $f_s^m(c)$ defined and $b \leqq c$, and $D(f_s^m(c)) \geqq 1 - (r+1)/n$, and set $g^n(b) = f_s^m(c)$.

Clearly $u_n = \{b \mid b \in (a_n^1 \cup \cdots \cup a_n^n), a \in u_{n-1}\}$ is a set of disjoint $b \in B$, with each $b \leqq$ some $a \in u_{n-1}$. We shall see now that u_n has supremum in B equal to 1. Suppose, if possible that for some $a \in u_{n-1}$, $\sup(b \mid b \in (a_n^1 \cup \cdots \cup a_n^n)) = \bar{a} \neq a$; then $a - \bar{a} \neq 0$. Then for any s, and any $m \geqq n(s)$ there is some c for which $f_s^m(c)$ is defined and $c \cap (a - \bar{a}) \neq 0$. Then this $b = c \cap (a - \bar{a})$ could be adjoined to a_n^n, contradicting the maximality of a_n^n. Thus $\sup u_n \geqq a$ for every $a \in u_{n-1}$, hence $\sup u_n = 1$, so u_n is a partition of B. Clearly g_u^n is a u_n-function.

Next we shall see that $\{g^n\}$ is fundamental. Indeed, if $g^n(b)$ and $g^m(c)$ are defined with $b \cap c \neq 0$, $n > m$, then $b \leqq c$, $D(g^n(b) \cup g^m(c)) \leqq D(g^m(c)) + 1/m$ since each a_n^r is maximal, and $D(g^n(b)) \geqq D(g^m(c)) - 1/m \geqq D(g^n(b) \cup g^m(c)) - 2/m$. **Hence**

$$\begin{aligned} D(g^n(b) \cup g^m(c)) &- D(g^n(b) \cap g^m(c)) \\ &= 2D(g^n(b) \cup g^m(c)) - D(g^n(b)) - D(g^m(c)) \\ &\leqq \frac{1}{m} + \frac{2}{m} = \frac{3}{m}. \end{aligned}$$

So $d(g^n, g^m) \to 0$ as $n, m \to \infty$. This means that $g = \{g^n\}$ is fundamental.

Next we show that $g \geqq f_s$ for each s. We need only show that $d(g \cup f_s, g) = 0$; hence we need only show that $d(g^m \cup f_s^{m(s)}, g^m) \to 0$ as $m \to \infty$.

Suppose $g^m(b)$ and $f_s^{m(s)}(c)$ are defined for some $b \cap c \neq 0$. Then

$D(f_s^{m(s)}(c) \cup g^m(b)) \leqq D(g^m(b)) + 2/m$ so $d(g^m \cup f_s^{m(s)}, g^m) \leqq 2/m$ and $\to 0$ as $m \to \infty$, as required.

Finally, we shall show that if h is fundamental and $h \geqq f_s$ for all s then $h \geqq g$. Suppose $h = \{h^n\}$ and choose $n(h)$ so that $d(h^m, h^p) < 1/n$ for all $m, p \geqq n(h)$. Then $d(h^m \cup f_s^p, h^m) \leqq 2/n$ if $m \geqq n(h)$ and $p \geqq n(s)$. Hence $d(h^m \cup g^n, h^m) \leqq 2/n$ if $m \geqq n(h)$ since each $g^n(b) = f_s^p(c)$ for some s and some $p \geqq n(s)$ and some $c \geqq b$. Hence $d(h^m \cup g^m, h^m) \to 0$ as $m \to \infty$. Thus $h \geqq g$ as required.

This completes the proof that $B(L)$ is complete.

LEMMA 4. *$L(B)$ is ascending-continuous.*

Proof. We need to show that if $f_\alpha \leqq f_\beta$ for all $\alpha < \beta < \Omega$ for some limit ordinal Ω, then $f \cap \bigcup(f_\alpha \mid \alpha < \Omega) = \bigcup(f \cap f_\alpha \mid \alpha < \Omega)$. Since $L(B)$ is complemented and modular it is sufficient to show that if $f \cap f_\alpha = 0$ for each α then $f \cap \bigcup(f_\alpha \mid \alpha \in I) = 0$.

Let $g = \bigcup(f_\alpha \mid \alpha \in I)$ be formed as in Lemma 3. Then since $f \cap f_\alpha = 0$, it follows for every α that $D(f^m(b) \cap f_\alpha^m(c)) < 2/n$ if $m \geqq n(f)$ and $m \geqq n(\alpha)$, where $n(f)$ is chosen so that $d(f^m, f^p) < 1/n$ if $m, p \geqq n(f)$, and $n(\alpha)$ is chosen so that $d(f_\alpha^m, f_\alpha^p) < 1/n$ if $m,p \geqq n(\alpha)$, and b,c are elements $\in B$ such that $f^m(b), f_\alpha^m(c)$ are defined and $b \cap c \neq 0$.

Now if $g^n(a)$ is defined then $g^n(a) = f_\alpha^m(c)$ for some α, some $m \geqq n(\alpha)$ and some c for which $a \leqq c$ and $f_\alpha^m(c)$ is defined.

Hence $d(f^m \cap g^m, 0) < 2/n$ if $m \geqq n(f)$ and $m \geqq n$, and so $\to 0$ as $m \to \infty$. Hence $d(f \cap g, 0) = 0$ so $f \cap g = 0$. This proves Lemma 4.

LEMMA 5. *$L(B)$ is descending-continuous.*

Proof. If $\{f_\alpha \mid \alpha \in I\}$ are given, we form $g = \bigcap(f_\alpha \mid \alpha \in I)$ by a procedure dual to that used in Lemma 3. Then Lemma 5 can be verified by an argument dual to that used in Lemma 4.

THEOREM 1. *$L(B)$ is a von Neumann geometry.*

Proof. This is a restatement of Lemmas 1–5.

REMARK 1. If L satisfies a chain condition, that is, L is the direct sum of a finite number of discrete (=finite dimensional irreducible projective) geometries then the proof of Theorem 1 can be simplified; in this case $B(L)$ coincides with the set of all partition functions.

REMARK 2. If $D(a) > 0$ for $a \neq 0$, $B(L)$ contains a sublattice $\bar{L}$ which is lattice isomorphic to L namely the constant sequences $f_a = \{f_a^n\}$ with $f_a^n(b)$ defined for $b = 1$ and $f_a^n(1) = a \in L$, for all n. The mapping $f_a \leftrightarrow a$ is a lattice isomorphism.

REMARK 3. Let B_0 be the Boolean algebra consisting of 0,1 only and suppose $D(a) > 0$ if $a \neq 0$, so $B_0(L)$ is a von Neumann geometry containing $\bar{L}$ (isomorphic to L) as a sublattice. Then $B_0(L) = \bar{L}$ if and only if L is a von Neumann geometry and $D(\bigcup(a_\alpha \mid \alpha < \Omega) = \sup(D(a_\alpha) \mid \alpha < \Omega)$ whenever $a_\alpha \leqq a_\beta$ for all $\alpha \leqq \beta < \Omega$.

REMARK 4. If L is a von Neumann geometry and $D(a) > 0$ for $a \neq 0$ and $D(\bigcup(a_\alpha \mid \alpha < \Omega)) = \sup(D(a_\alpha) \mid \alpha < \Omega)$ whenever $a_\alpha \leqq a_\beta$ for all $\alpha \leqq \beta < \Omega$, then Z the centre of L is a complete Boolean algebra isomorphic to a finite measure algebra (then Z cannot contain a noncountable set of nonzero disjoint elements).

REMARK 5. We could construct $B(L)$ in terms of an arbitrary family of given dimension functions D_i (in place of one function D) but then to obtain a von Neumann geometry $B(L)$ we would need to use fundamental filters f (in place of fundamental sequences). We omit the details.

We shall call a fundamental sequence $\{f^n\}$ *distributive* if whenever $f^n(b)$ is defined its value is distributive in L ($z \in L$ is called distributive in L if $z \cap (a \cup b) = (z \cap a) \cup (z \cap b)$ for all $a, b \in L$). It is clear that the distributive fundamental sequences form a sublattice of $B(L)$ which is isomorphic to (and henceforth will be identified with) $B(Z) \equiv B(D,Z)$ where Z is the Boolean algebra of all distributive elements in L (in any complemented modular lattice, the set of distributive elements coincides with the centre of the lattice).

LEMMA 6. *f is distributive in $B(L)$ if $(*) f$ is equivalent to an element in $B(Z)$. If L is a von Neumann geometry then f is distributive in $B(L)$ if and only if $(*)$ holds; the centre of $B(L)$ is $B(Z)$.*

Proof. The first part of Lemma 6 follows from the definitions of $f \cup g, f \cap g$.

On the other hand, if $f = \{f^n\}$ is distributive in $B(L)$ then for each n let $q_n = \sup\{\inf\{d(f^n(b), z) \mid z \in Z\} \mid \text{all } b \text{ for which } f^n(b) \text{ is defined}\}$, let $z(b) \in Z$ be chosen so that $d(f^n(b), z(b)) < q_n + 1/n$ and let $\bar{f}^n(b)$ be defined, $= z(b)$ for each b for which $f^n(b)$ is defined.

If $q_n \to 0$ as $n \to \infty$ it will follow that $\bar{f} = \{\bar{f}^n\}$ is fundamental, $\in B(Z)$ and $f \equiv \bar{f}$, as desired.

Thus to prove Lemma 6, we may assume that for some fixed $\varepsilon > 0$: $q_n > \varepsilon$ for an infinite number of n and we need only derive a contradiction (assuming that L is a von Neumann geometry). Now for some n_0, and for some b_0 for which $f^{n_0}(b_0)$ is defined: (i) $d(f^{n_0}(b_0), z) > \varepsilon$ for all $z \in Z$ and (ii) for all $n \geqq n_0$ the c_n for which $f^n(c_n)$ is defined and $c_n \cap b_0 \neq 0$ satisfy $d(f^n(c_n), f^{n_0}(b_0)) < \varepsilon/4$.

But for any a,c in L, the constant sequences $g = \{g^n\}$, $h = \{h^n\}$ with $g^n(1) = a$, $h^n(1) = c$ for all n, are fundamental; since f is distributive in $B(L)$ hence $d((f \cap (g \cup h)), (f \cap g) \cup (f \cap h)) = 0$, so

$$d((f^n(c_n) \cap (a \cup c)), ((f^n(c_n) \cap a) \cup (f^n(c_n) \cap c))) \to 0$$

as $n \to \infty$. Hence

$$d(f^{n_0}(b_0) \cap (a \cup c), \quad (f^{n_0}(b_0) \cap a) \cup (f^{n_0}(b_0) \cap c)) < \frac{\varepsilon}{2}.$$

Thus to obtain the desired contradiction we need only show: if $b \in L$ with $d(b,z) > \varepsilon$ for all $z \in Z$ then there exists $a, c \in L$ such that

$$d(b \cap (a \cup c), (b \cap a) \cup (b \cap c)) > \frac{\varepsilon}{2}.$$

This is shown in the following lemma, which completes the proof of Lemma 6.

LEMMA 7. *Suppose that L is a von Neumann geometry with a dimension function $D(a)$ such that $0 \leqq D(a) \leqq 1$ for all $a \in L$, $D(0) = 0$, $D(1) = 1$ and $D(a \cup b) + D(a \cap b) = D(a) + D(b)$ for all a, b in L. Then for any b in L,*

$$\inf\{(D(b \cup z) - D(b \cap z)) \mid z \in Z\} \leqq D(b \cap (a \cup c)) - D((b \cap a) \cup (b \cap c))$$

for some $a, c \in L$.

Proof. It is sufficient to exhibit $a, c \in L$, $z_0 \in Z$ such that

$$D(b \cup z_0) - D(b \cap z_0) = D(b \cap (a \cup c)) - D((b \cap a) \cup (b \cap c)).$$

For this purpose let h be any complement in L of b. Apply von Neumann [6, Part III, Theorem 2.7] to b and h. Then there exist disjoint elements $e_1, e_2 \in Z$ such that

$$b = b_1 \cup b_2 \text{ with } b_1 = b \cap e_1, b_2 = b \cap e_2,$$

$$h = h_1 \cup h_2 \text{ with } h_1 = h \cap e_1, h_2 = h \cap e_2,$$

and h_1 is perspective to some $\bar{b}_1 \leqq b_1$ and b_2 is perspective to some $\bar{h}_2 \leqq h_2$. Hence for suitable x, y in L,

$$h_1 \cup x = \bar{b}_1 \cup x = h_1 \cup \bar{b}_1; \quad h_1 \cap x = \bar{b}_1 \cap x = 0;$$

$$\bar{h}_2 \cup y = b_2 \cup y = \bar{h}_2 \cup b_2; \quad \bar{h}_2 \cap y = b_2 \cap y = 0.$$

Set $a = h_1 \cup \bar{h}_2, c = x \cup y$. Then

$$\begin{aligned} D(b \cap (a \cup c)) &- D((b \cap a) \cup (b \cap c)) \\ &= D(b \cap (h_1 \cup \bar{h}_2 \cup x \cup y)) - D((b \cap (h_1 \cup \bar{h}_2)) \cup (b \cap (x \cup y))) \\ &= D(b_1 \cap (h_1 \cup x)) + D(b_2 \cap (\bar{h}_2 \cup y)) - 0 \\ &= D(\bar{b}_1) + D(b_2) = D(h_1) + D(b_2) \\ &= D(e_1) - D(b_1) + D(b_2) \\ &= D(b \cup e_1) - D(b \cap e_1). \end{aligned}$$

Since e_1 is in Z, Lemma 7 (and hence also Lemma 6) is established.

THEOREM 2. *If L is an irreducible von Neumann geometry and B is a complete Boolean algebra, the centre of $B(L)$ is isomorphic to B.*

Proof. Suppose that $b \in B$ and let $f_b = \{f_b^n\}$ be the constant sequence with $f_b^n(b)$, $f_b^n(1 - b)$ defined, $= 1, 0$, respectively, for all n. Then the mapping $b \leftrightarrow f_b$ is clearly a (1,1) lattice isomorphism of B onto a sublattice of the centre of $B(L)$ since $d(f_b,f_c) = D(1) \neq 0$ if $b \neq c$.

On the other hand, suppose f is in the centre of $B(L)$. We may suppose $f = \{f^n\}$ with $\{f^n\}$ a distributive sequence. Now for any n, if $f^n(b)$ is defined, we must have $f^n(b) = 0$ or 1 since L has centre consisting of 0,1 only. Let $b^n = \bigcup(b \mid f^n(b) = 1)$. Then $d(f^n,f^m) \geqq D(1)$ if $b^n \neq b^m$. Hence for some n_0, b^n is constant, $= b$ say, for all $n \geqq n_0$ and f is equivalent to the constant sequence f_b.

This proves Theorem 2.

4. **The Iwamura local components of** $B(L)$. Iwamura [3; 4], defined local components for any von Neumann geometry L with centre Z in (essentially) the following way.

First, write: $|f| \leqq 1/r$ to mean that r is a positive integer, $f \in L$ and there exist independent elements $f = f_1, f_2, \cdots, f_r$ in L such that each f_i is perspective to f.

Next, let p be a maximal dual ideal of Z. This means: $p \subset Z$ and p has the following properties:

(i) $b \in p, c \in Z$, $c \geqq b$ together imply $c \in p$;

(ii) $b \in p, c \in p$ together imply $b \cap c \in p$;

(iii) $0 \notin p$;

(iv) p is maximal with properties (i), (ii), (iii).

Next, define $f \leqq g$ (at p) to mean: $f, g \in L$ and for any relative complement $h = [(f \cup g) - g]$, and for each integer $r > 0$, there exists some b_r in p such that $|b_r \cap h| \leqq 1/r$. Define $f \equiv g$ (at p) to mean: $f \leqq g$ (at p) and $g \leqq f$ (at p) both hold.

Then as Iwamura showed, the relation "$\equiv$ (at p)" is an equivalence relation in L and if f is identified with its equivalence class, the set of equivalence classes is ordered under the relation "$\leqq$ (at p)" and is an irreducible von Neumann geometry, denoted L/p. We shall call *L/p the Iwamura local component of L at p.*

Throughout the rest of this section we shall assume that L is an irreducible von Neumann geometry, that D denotes its (unique) dimension function, that B is a complete Boolean algebra and that p is a maximal dual ideal of B. For each $c \in L$, $f_c = \{f_c^n\}$ will denote the constant sequence with $f_c^n(1)$ defined and equal to c for all n.

We shall prove:

THEOREM 3. *If L is an irreducible von Neumann geometry and B is a complete Boolean algebra then L is isomorphic to the subgeometry of all f_c in $B(L)$ (at p) under the mapping $c \to f_c$.*

LEMMA 8. *With the hypotheses of Theorem 3, if $f_a \leqq f_c$ (at p) then $a \leqq c$.*

Proof. Let $e = [(a \cup c) - c]$. Then $f_e = [(f_a \cup f_c) - f_c]$. For every integer $r > 0$ there exists some $b \in p$ for which $|b \cap f_e| < 1/r$. This means: there are elements $f_1 = b \cap f_e, f_2, \cdots, f_r$ independent in $B(L)$ and elements $g_2, \cdots, g_r$ in $B(L)$ such that $g_j \cup f_1 = g_j \cup f_j = 1$, $g_j \cap f_1 = g_j \cap f_j = 0$ (in $B(L)$) for $j \geqq 2$.

We may, without loss of generality, suppose that for each n there is a partition u_n and $f_j = \{f_j^n\}$, $g_j = \{g_j^n\}$ with u_n-functions f_j^n and g_j^n.

For each n choose an element $b_n \in u_n$. Then as $n \to \infty$:

$$D(g_j^n(b_n) \cup e) \to 1, \; D(g_j^n(b_n) \cup f_j^n(b_n)) \to 1,$$

$$D(g_j^n(b_n) \cap e) \to 0, \; D(g_j^n(b_n) \cap f_j^n(b_n)) \to 0,$$

and for $2 \leqq j \leqq r$,

$$D\left(f_j^n(b_n) \cap \left(\bigcup_{i=2}^{i-1} f_i^n(b_n) \cup e\right)\right) \to 0.$$

It follows that for $2 \leqq j \leqq r$,

$$D(f_j^n(b_n)) \to D(e) \text{ as } n \to \infty,$$

and

$$D(e \cup f_2^n(b_n) \cup \cdots \cup f_r^n(b_n)) \to rD(e) \text{ as } n \to \infty.$$

Since $D(a) \leqq 1$ for all $a \in L$, $rD(e) \leqq 1$, $D(e) \leqq 1/r$.

Since $D(e) \leqq 1/r$ for all $r > 0$, $e = 0$ and so $a \leqq c$ as stated.

Proof of Theorem 3. Obviously $a \leqq b$ implies $f_a \leqq f_b$ (at p); Lemma 8 shows that $f_a \leqq f_b$ (at p) implies $a \leqq b$. It follows that the mapping $c \to f_c$ is an isomorphism of L and the subgeometry of $B(L)/p$ consisting of the elements of the form f_c.

We shall call an irreducible von Neumann geometry L *compact* if for each integer n there exist a finite set of elements $a_1, \cdots, a_m \in L$ $(m = m(n))$ such that for each $b \in L$: $\min(D(a_i \cup b) - D(a_i \cap b) \mid i = 1, \cdots, m) < 1/n$; in other words, L is compact in the metric topology

$$d(a, b) = D(a \cup b) - D(a \cap b).$$

It is easy to see that L is compact if and only if for each real number α $(0 \leqq \alpha \leqq 1)$ the set of elements of dimension α is compact (or empty); if L is a discrete (that is, projective finite dimensional) geometry, this condition is equivalent to: the number of elements in L is finite.

We shall prove:

Theorem 4. *If L is an irreducible von Neumann geometry and B is a complete Boolean algebra, the Iwamura local components of $B(L)$ are all isomorphic to L if either the number of elements in B is finite or L is compact.*

Lemma 9 (von Neumann [5, p. 107]). *Every irreducible von Neumann geometry L is a complete metric space under the metric: $d(a,b) = D - (a \cup b) - D(a \cap b)$.*

Proof. Suppose that $d(a_n, a_m) \to 0$ as $n, m \to \infty$. Choose a subsequence $a_{n(m)}$, with $n(m) < n(m+1)$ for all $m = 1, 2, \cdots$ such that $d(a_{n(m)}, a_{n(m+1)}) < 1/m^2$. Then $a = \bigcap_{m=1}^{\infty}(\bigcup_{r=m}^{\infty} a_{n(r)})$ satisfies: $d(a, a_{n(m)}) \to 0$ as $m \to \infty$ and hence $d(a, a_n) \to 0$ as $n \to \infty$.

LEMMA 10. *Suppose that L is an irreducible von Neumann geometry and $a, b_1, c \in L$ with $b_1 \leqq a \cup c, a \cap c = b_1 \cap c = a \cap b_1 = 0$ and $D(a) = D(c)$. Then for some $b \geqq b_1$: $a \cup b = b \cup c = a \cup c$ and $a \cap b = b \cap c = 0$.*

Proof. Let $a_1 = a \cap (b_1 \cup c)$, $c_1 = c \cap (b_1 \cup a)$. Then

$$a_1 \cup b_1 = (a \cup b_1) \cap (c \cup b_1) = c_1 \cup b_1$$

$$= a_1 \cup c_1, a_1 \cap b_1 = b_1 \cap c_1 = c_1 \cap a_1 = 0.$$

Now let $a_2 = [a - a_1], c_2 = [c - c_1]$. Then $D(a_2) = D(a) - D(a_1) = D(c) - D(c_1) = D(c_2)$, $a_2 \cap c_2 = 0$ and $(a_2 \cup c_2) \cap (a_1 \cup c_1) = 0$. Hence for some $b_2 \leqq a_2 \cup c_2$ we have $a_2 \cup b_2 = c_2 \cup b_2 = a_2 \cup c_2$ and $a_2 \cap b_2 = c_2 \cap b_2 = 0$.

It is easily seen that $b = b_1 \cup b_2$ satisfies the requirements of Lemma 10.

LEMMA 11. *Suppose that L is an irreducible von Neumann geometry, B is a Boolean algebra, $f = \{f^n\}$ is in $B(L)$, and for some $r \geqq 2$, $D(f^n(b)) \leqq 1/r$ for all n and b for which $f^n(b)$ is defined. Then $|f| \leqq 1/r$.*

Proof. We may suppose each f^n is a u_n-function with partitions u_n such that $b \in u_n$, $c \in u_{n+1}$, $b \cap c \neq 0$ imply $c \leqq b$. By using a suitable subsequence of the f^n we may also suppose that $\sum_{n=1}^{\infty} d(f^n, f^{n+1}) < \infty$. Denote f as f_1.

We shall show that for some fixed constant $k < \infty$, we can define, for each n and each $b \in u_n$, values in L, $f_j^n(b)$, $g_j^n(b)$, $j = 2, \cdots, r$ so that

$$f_1^n(b), \cdots, f_r^n(b) \text{ are independent in } L,$$

$$(\mathrm{P}_n) \qquad g_j^n(b) \cup f_1^n(b) = g_j^n(b) \cup f_j^n(b) = f_1^n(b) \cup f_j^n(b),$$

$$g_j^n(b) \cap f_1^n(b) = g_j^n(b) \cap f_j^n(b) = 0,$$

and so that if $n > 1$ and $b \in u_{n-1}$, $c \in u_n$ and $c \leqq b$, then

$$d(g_j^{n-1}(b), g_j^n(c)) \leqq kd(f^n, f^{n+1}),$$

$$(\mathrm{Q}_m) \qquad d(f_j^{n-1}(b), f_j^n(c)) \leqq kd(f^n, f^{n+1}).$$

It will follow that $g_j = \{g_j^n\}$, $f_j = \{f_j^n\}$, $j = 2, \cdots, r$, are all in $B(L)$, that $f_1, \cdots, f_r$ are independent in $B(L)$ and for $j \geqq 2$, f_j is perspective to f_1. This will show that $|f| \leqq 1/r$ as required.

We define $f_j^n(b)$, $g_j^n(b)$ for all $b \in u_n$ by induction on n. For $n = 1$, we choose arbitrarily $f_j^1(b), g_j^1(b)$ for all $b \in u_1$, to satisfy (P_1). This is possible since $D(f_1^1(b)) \leqq 1/r$ for all $b \in u_1$.

Now suppose for some $n \geqq 1$ that $g_j^n(b)$, $f_j^n(b)$ are defined for all $b \in u_n$ and satisfy (P_n). Then we define $g_j^{n+1}(c), f_j^{n+1}(c)$ for all $c \in u_{n+1}$, and $j = 2, \cdots, s, \cdots, r$ by induction on s as follows, so as to satisfy (P_{n+1}) and (Q_{n+1}).

Let $\bar{g}_2^{n+1}(c) = g_2^n(b) \cap f_1^{n+1}(c)$. If $D(g_2^n b)) - D(\bar{g}_2^{n+1}(c)) \geqq D(f_1^{n+1}(c))$ choose $g_2^{n+1}(c)$ arbitrarily $\leqq g_2^n(b)$ but with $g_2^{n+1}(c) \cap f_1^{n+1}(c) = 0$ and

$$D(g_2^{n+1}(c) = D(f_1^{n+1}(c)).$$

Otherwise, choose $g_2^{n+1}(c) = [g_2^n(b) - \bar{g}_2^{n+1}(c)] \cup h_2^{n+1}(c)$ where $h_2^{n+1}(c)$ is any element in L with $h_2^{n+1}(c) \cap (g_2^n(b) \cup f_1^{n+1}(c)) = 0$ and $D(g_2^{n+1}(c)) = D(f_1^{n+1}(c))$. Such an element $h_2^{n+1}(c)$ exists since in the present case, $D(g_2^n(b) \cup f_1^{n+1}(c)) \leqq 2D(f_1^{n+1}(c))$.

Now $g_2^{n+1}(c) \cap f_1^{n+1}(c) = 0$. Let $\bar{f}_2^{n+1}(c) = f_2^n(b) \cap (g_2^{n+1}(c) \cup f_1^{n+1}(c))$ and let $(\bar{f}_2')^{n+1}(c) = (\bar{f}_2^{n+1}(c) \cap g_2^{n+1}(c)) \cup (\bar{f}_2^{n+1}(c) \cap f_1^{n+1}(c))$.

Then $D(\bar{f}_2^{n+1})(c)) - D(\bar{f}_2')^{n+1}(c)) \leqq D(f_1^{n+1}(c))$. Choose

$$f_2^{n+1}(c) \leqq g_2^{n+1}(c) \cup f_1^{n+1}(c) \text{ and } \geqq {}_L\bar{f}_2^{n+1}(c) - (\bar{f}_2')^{n+1}(c)]$$

and so that

$$f_2^{n+1}(c) \cap g_2^{n+1}(c) = f_2^{n+1}(c) \cap f_1^{n+1}(c) = 0$$

and $f_2^{n+1}(c) \cup g_2^{n+1}(c) = f_1^{n+1}(c) \cup g_2^{n+1}(c) = f_2^{n+1}(c) \cup f_1^{n+1}(c)$. This is possible because of Lemma 10.

Now suppose for some $s = 2, \cdots, r-1$ that $f_j^{n+1}(c), g_j^{n+1}(c)$ have been defined for all $2 \leqq j \leqq s$ so that: $f_1^{n+1}(c), \cdots, f_s^{n+1}(c)$ are independent, $f_1^{n+1}(c) \cup g_j^{n+1}(c) = f_j^{n+1}(c) \cup g_j^{n+1}(c) = f_1^{n+1}(c) \cup g_j^{n+1}(c)$ and $f_1^{n+1}(c) \cap g_j^{n+1}(c) = f_j^{n+1}(c) \cap g_j^{n+1}(c) = 0$. Then define $g_{s+1}^{n+1}(c)$, $f_{s+1}^{n+1}(c)$ as follows.

Let $\bar{g}_{s+1}^{n+1}(c) = g_{s+1}^n(b) \cap (f_1^{n+1}(c) \cup \cdots \cup f_s^{n+1}(c))$. If $D(g_{s+1}^n(b)) - D(\bar{g}_{s+1}^{n+1}(c)) \geqq D(f_1{}^{+1}(c))$, choose $g_{s+1}^{n+1}(c) \leqq g_2^n(b)$ but with $g_{s+1}^{n+1}(c) \cap \bar{g}_{s+1}^{n+1}(c) = 0$ and $D(g_{s+1}^{n+1}(c)) = D(f_1^{n+1}(c))$. Otherwise, choose

$$g_{s+1}^{n+1}(c) = [g_{s+1}^n(b) - \bar{g}_{s+1}^{n+1}(c)] \cup h_{s+1}^{n+1}(c)$$

with $h_{s+1}^{n+1}(c) \cap (g_{s+1}^n(b) \cup f_1^{n+1}(c) \cup \cdots \cup f_s^{n+1}(c)) = 0$ and $D(g_{s+1}^{n+1}(c)) = D(f_1^{n+1}(c))$. Such an element $h_{s+1}^{n+1}(c)$ exists since in the present case,

$$D(g_{s+1}^n(b) \cup f_1^{n+1}(c) \cup \cdots \cup f_s^{n+1}(c)) \leqq (s+1)/r \leqq 1.$$

Now $\{g_{s+1}^{n+1}(c), f_1^{n+1}(c), \cdots, f_s^{n+1}(c)\}$ are independent in L. Let $\bar{f}_{s+1}^{n+1}(c) = f_{s+1}^n(b) \cap (g_{s+1}^{n+1}(c) \cup f_1^{n+1}(c))$ and let $(\bar{f}_{s+1}')^{n+1}(c) = (\bar{f}_{s+1}^{n+1}(c) \cap g_{s+1}^{n+1}(c)) \cup (\bar{f}_{s+1}^{n+1}(c) \cap f_1^{n+1}(c)$.

Then $D(\bar{f}_{s+1}^{n+1}(c)) - D((\bar{f}_{s+1}')^{n+1}(c)) \leqq D(f_1^{n+1}(c))$. Choose $f_{s+1}^{n+1}(c) \leqq g_{s+1}^{n+}(c) \cup f_1{}^{n+1}(c)$ and $\geqq [\bar{f}_{s+1}^{n+1}(c) - (\bar{f}'_{s+1})^{n+1}(c)]$ and so that $f_{s+1}^{n+1}(c) \cap g_{s+1}^{n+1}(c) = f_{s+1}^{n+1}(c) \cap f_1^{n+1}(c) = 0$ and $f_{s+1}^{n+1}(c) \cup g_{s+1}^{n+1}(c) = f_1^{n+1}(c) \cup g_{s+1}^{n+1}(c) = f_{s+1}^{n+1}(c) \cup f_1^{n+1}(c)$. This is possible because of Lemma 10.

Now by induction on s, (P_{n+1}) will be satisfied. As for (Q_{n+1}), we calculate: either $d(g_2^n(b), g_2^{n+1}(c)) \leqq d(f_1^n(b), f_1^{n+1}(c))$ or (since $D(\bar{g}_2^{n+1}(c)) \leqq d(f_1^n(b), f_1^{+1}(c))$) $d(g_2^n(b), g_2^{n+1}(c)) \leqq 2d(f_1^n(b), f_1^{n+1}(c))$.

Next,

$$D(f_2^n(b)) - D(f_2^{n+1}(c)) \leqq d(g_2^n(b), g_2^{n+1}(c)) + d(f_1^n(b), f_1^{n+1}(c)) \leqq 3d(f_1^n(b), f_1^{n+1}(c)),$$

$$D((f_2')^{n+1}(c)) \leqq 3d(f_1^n(b), f_1^{n+1}(c)),$$

hence $d(f_2^n(b), f_2^{n+1}(c)) \leqq 12d(f_1^n(b), f_1^{n+1}(c))$.

Similarly it can be verified that (Q_{n+1}) holds for some constant k which is independent of n.

Thus Lemma 11 holds.

LEMMA 12. *With the hypotheses of Theorem 4 suppose that $f = \{f^n\}$ is in $B(L)$, p is a maximal dual ideal of B, and for an infinite number of n: f^n is a a u_n-function and some $b_n \in u_n$ satisfies $b_n \in p$. Then for some $c \in L$, $f \equiv f_c$ (at p).*

Proof. Replacing $\{f^n\}$ by an equivalent sequence we may suppose that for each n: f^n is a u_n-function and $b_n \in u_n$, $b_n \in p$ hold.

Since $d(f^n(b_n), f^m(b_m)) \to 0$ as $n, m \to \infty$ it follows from Lemma 9 that $d(f^n(b_n), c) \to 0$ as $n \to \infty$ for some $c \in L$.

We shall prove, with this element c, that $f \equiv f_c$ (at p). As in the proof of Lemma 2 we may suppose that $g^n = [(f^n \cup f_c^n) - (f^n \cap f_c^n)]$ is chosen in such a way that $g = \{g^n\}$ satisfies $g = [(f \cup f_c) - (f \cap f_c)]$ in $B(L)$.

To show that $f \equiv f_c$ (at p) is equivalent to showing: for each $r > 0$ there exists some $b \in p$ for which $|g \cap b| < 1/r$.

Choose a fixed s so that $D(g^s(b_s)) < 1/2r$ and $d(g^s, g^n) < 1/2r$ for $n \geqq s$. Then $b_s \cap g = \{b_s \cap g^{n+s}\}$ satisfies the hypothesis of Lemma 11, hence $|b_s \cap g| < 1/r$. This proves Lemma 12.

COROLLARY. *If b_0 is an atom in B and p consists of all $b \in B$ with $b \geqq b_0$ then $B(L)/p$ is isomorphic to L under the mapping $c \to f_c$ of L onto the subgeometry of all f_c in $B(L)$ (at p).*

Proof. Assume given $f = \{f^n\}$ in $B(L)$ with f^n a u_n-function. Then for some (unique) $b_n \in u_n$, $b_n \geqq b_0$ and hence $b_n \in p$. Lemma 12 applies and shows that $f \equiv f_c$ for some $c \in L$. The Corollary now follows from Theorem 3.

Proof of Theorem 4. If B has only a finite number of elements, every maximal dual ideal p is determined by some atom, as in the hypothesis of the corollary to Lemma 12, and for this case, Theorem 4 follows from the corollary to Lemma 12.

Suppose now that L is compact and for each n, choose $a_1, \cdots, a_m$ $(m = m(n))$

in L so that for each c in $L, d(a_i, c) < 1/n$ for at least one i. Given any $f = \{f^n\}$ in $B(L)$ with f^n a u_n-function set, for each n,

$$b_i^n = \bigcup (b \mid f^n(b) \text{ is defined and } i \text{ is the first index for which } d(a_i, f^n(b)) < 1/n)$$

and define $g = \{g^n\}$ by $g^n(b_i^n) = a_i$ $(i = 1, \cdots, m)$. It is easy to see that $g = f$ in $B(L)$ and for every maximal dual ideal p and each n, one of the $b_i^n \in p$. Hence Lemma 12 applies andfor the case of compact L, Theorem 4 follows from Theorem 3.

This completes the proof of Theorem 4.

References

1. Garrett Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloq. Publ. Vol. 25, Amer. Math. Soc., Providence, R. I., 1948.

2. Israel Halperin, *Dimensionality in reducible geometries*, Ann. of Math. **40** (1939), 581–599.

3. Tsurane Iwamura, *On continuous geometries*. I, Japan. J. Math. **19** (1944), 57–71.

4. Fumitomo Maeda, *Kontinuierliche Geometrien*, Springer, Berlin, 1958.

5. John von Neumann, *Continuous geometry and examples of continuous geometries*, Proc. Nat. Acad. Sci. U. S. A. **22** (1936), 92–108.

6. ———, *Continuous geometry*, Princeton Univ. Press, Princeton, N. J., 1960.

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